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Response of Vibrating Systems with Perturbed Parameters

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A second-order perturbation theory is developed for the response of perturbed, undamped, nongyroscopic dynamical systems, where the perturbations arise from small changes in the system mass and stiffness coefficients. The solution is based upon the eigensolution of the unperturbed system and the perturbations in the mass and stiffness coefficients. The object is to obtain the eigensolution of the perturbed system without repeating the original computation cycle. This work is of particular importance for large-order systems for which an eigensolution analysis is costly and time-consuming.

Introduction

UNDAMPED nongyroscopic dynamical systems are characterized by real symmetric mass and stiffness matrices, where the mass matrix is positive definite and the stiffness matrix is either positive definite or positive semidefinite. Due to the special nature of such systems, the response can be obtained with relative ease when compared to the difficulty involved in obtaining the response of a general dynamical system. This advantage becomes more pronounced as the order of the system increases, although for very high-order systems the computational effort in obtaining the eigensolution can become costly.

One common problem in structural dynamics analysis is how to cope with changes in design, especially if the changes are not too great. In particular, the question arises as to whether it is necessary to repeat a costly analysis or it is possible to salvage some of the eigensolution results obtained earlier. Observing that small changes in a design are likely to result in small changes in the mass and stiffness matrices, without affecting their basic properties, one must conclude that the eigenvalues and eigenvectors of the redesigned system cannot be appreciably different from those of the original system. Regarding the differences in the mass and stiffness matrices between the original and redesigned system as small perturbations, a second-order theory is developed for the computation of the eigensolution of the redesigned system based on the original eigensolution. The eigensolution obtained by this perturbation method is then used to determine the response of the redesigned system to arbitrary excitation.

In an earlier paper,[1] Chen and Wada have addressed the same problem, carrying the analysis through first-order. The present endeavor differs from that of Chen and Wada in the following respects. The present analysis considers an arbitrary excitation, whereas that of Chen and Wada considers three special types. The present work produces the dynamic response in two stages. The first stage entails production of the perturbed eigensolutions corresponding to the perturbed mass and stiffness coefficients. Then, the second stage is a modal analysis wherein the equations of motion are uncoupled via the perturbed eigenvectors. This two-stage approach seems to provide a more direct and general formulation and solution of the problem.

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Response of Undamped Systems

The equations of motion of an undamped, nongyroscopic, n-degree-of-freedom system can be written in the matrix form[2]

$$M\ddot{q}(t)+Kq(t)=Q(t) \tag{1}$$

where M and K are real symmetric inertia and stiffness matrices, $q(t)$ is an n-vector of generalized coordinates, and $Q(t)$ is an n-vector of generalized forces. The matrix M is positive definite by definition, and the matrix K is assumed to be positive definite.

The solution of Eq. (1) can be conveniently obtained by modal analysis. The procedure is well-documented (see for example Ref. 2) and will not be repeated here. Instead, for future reference, we shall give a brief summary of its salient features. The eigenvalue problem associated with Eq. (1) is

$$\lambda_i Mq_i=Kq_i \qquad (i=1,2,...,n) \tag{2}$$

where λ_i and q_i are eigenvalues and eigenvectors, respectively. Because M and K are real, symmetric, and positive definite, all the eigenvalues are real and positive. They can be written as

$$\lambda_i=\omega_i^2 \qquad (i=1,2,...,n) \tag{3}$$

where the ω_i are the natural frequencies of the system. We assume the eigenvalues to be distinct, and furthermore, that no two eigenvalues are equal or nearly equal. The eigenvectors are mutually orthogonal. If they are normalized so that $q_i^TMq_i=1$ $(i=1,2,...,n)$, then the orthonormality of the eigenvectors can be expressed as

$$q_j^TMq_i=\delta_{ij} \qquad (i,j=1,2,...,n) \tag{4}$$

where δ_{ij} is the Kronecker delta. From Eq. (2), it follows immediately that

$$q_j^TKq_i=\lambda_i\delta_{ij} \qquad (i,j=1,2,...,n) \tag{5}$$

Introducing the modal matrix,

$$P=[q_1q_2...q_n] \tag{6}$$

Eqs. (4) and (5) can be written in the condensed form

$$P^TMP=I \tag{7}$$

$$P^TKP=\Lambda \tag{8}$$

where I is the identity matrix, and Λ is the diagonal matrix of eigenvalues

$$\Lambda = \text{diag}\ \lambda_i \qquad (i=1,2,...,n) \tag{9}$$

The objective of a modal analysis is to uncouple the equations of motion of the system, Eq. (1). To this end, we use a linear transformation

$$q(t) = \sum_{k=1}^{n} q_k \eta_k(t) = P\eta(t) \tag{10}$$

where $\eta(t)$ is an n-vector of independent generalized coordinates. Introducing Eq. (10) into Eq. (1), premultiplying the result by P^T, and utilizing Eqs. (7) and (8), one obtains

$$\ddot{\eta}(t) + \Lambda\eta(t) = N(t) \tag{11}$$

where

$$N(t) = P^T Q(t) \tag{12}$$

Any one of Eqs. (11) can be written as

$$\ddot{\eta}_k(t) + \omega_k^2 \eta_k(t) = N_k(t) \qquad (k=1,2,...,n) \tag{13}$$

The solution of Eqs. (13) can be written in terms of the convolution integral as

$$\eta_k(t) = \eta_k(0)\cos\omega_k t + \frac{\dot{\eta}_k(0)}{\omega_k}\sin\omega_k t + \frac{1}{\omega_k}\int_0^t N_k(\tau)\sin\omega_k(t-\tau)\,d\tau \qquad (k=1,2,...,n) \tag{14}$$

where

$$\eta_k(0) = q_k^T M q(0) \quad \dot{\eta}_k(0) = q_k^T M \dot{q}(0) \quad (k=1,2,...,n) \tag{15}$$

The formal solution is obtained by introducing Eqs. (12), (14), and (15) into Eq. (10). The result is

$$q(t) = \sum_{k=1}^{n} q_k q_k^T M\left(q(0)\cos\omega_k t + \frac{\dot{q}(0)}{\omega_k}\sin\omega_k t\right) + \sum_{k=1}^{n} q_k q_k^T \int_0^t Q(\tau)\sin\omega_k(t-\tau)\,d\tau \tag{16}$$

Eigenvalue Problem for the Redesigned System

Let us suppose that the original system is defined by the mass and stiffness matrices M_0 and K_0, respectively, and the redesigned system by the matrices

$$M = M_0 + M_1 \qquad K = K_0 + K_1 \tag{17}$$

where M_0 and K_0 are real, symmetric, positive definite matrices, and where M_1 and K_1 are real and symmetric. One should note that while M_1 and K_1 possess no inherent definiteness, they must be such that the sums $M_0 + M_1$ and $K_0 + K_1$ are positive definite. Now, if the redesigned system does not differ greatly from the original system, then the matrices M_1 and K_1 can be regarded as small when compared to M_0 and K_0, respectively. Our object then is to obtain solutions of the eigenvalue problem

$$\lambda_i(M_0 + M_1)q_i = (K_0 + K_1)q_i \qquad (i=1,2,...,n) \tag{18}$$

based upon the eigensolutions λ_{0i}, q_{0i} of the eigenvalue problem

$$\lambda_{0i} M_0 q_{0i} = K_0 q_{0i} \qquad (i=1,2,...,n) \tag{19}$$

Let us first cast Eq. (19) into a standard form. Because M_0 is a real, symmetric, positive definite matrix, we can use the Cholesky decomposition and write

$$M_0 = L_0 L_0^T \tag{20}$$

Introducing the linear transformation

$$L_0^T q_{0i} = u_{0i} \qquad q_{0i} = L_0^{-T} u_{0i} \qquad (i=1,2,...,n) \tag{21}$$

in which $L_0^{-T} = (L_0^{-1})^T = (L_0^T)^{-1}$, and premultiplying the result by L_0^{-1}, Eq. (19) can be written in the standard form

$$\lambda_{0i} u_{0i} = A_0 u_{0i} \qquad (i=1,2,...,n) \tag{22}$$

where

$$A_0 \equiv L_0^{-1} K_0 L_0^{-T} \tag{23}$$

The eigenvectors u_{0i} are orthogonal and can be normalized so as to satisfy the orthonormality relations

$$u_{0j}^T u_{0i} = \delta_{ij} \qquad u_{0j}^T A_0 u_{0i} = \lambda_{0i}\delta_{ij} \qquad (i,j=1,2,...,n) \tag{24}$$

Let us return to Eq. (18) and consider the linear transformation

$$L_0^T q_i = u_i \qquad q_i = L_0^{-T} u_i \qquad (i=1,2,...,n) \tag{25}$$

Introducing Eqs. (20) and (25) into Eq. (18), we obtain

$$\lambda_i(I + B_1)u_i = (A_0 + A_1)u_i \qquad (i=1,2,...,n) \tag{26}$$

where

$$B_1 = L_0^{-1} M_1 L_0^{-T} \qquad A_1 = L_0^{-1} K_1 L_0^{-T} \tag{27}$$

Recognizing that the eigenvalue problem (18) is the same as the eigenvalue problem (2), and recalling Eqs. (4) and (5), the eigenvectors u_i must satisfy the orthonormality relations

$$u_j^T(I + B_1)u_i = \delta_{ij} \quad u_j^T(A_0 + A_1)u_i = \lambda_i\delta_{ij} \quad (i,j=1,2,...,n) \tag{28}$$

We note that the eigenvalue problem expressed by Eq. (26) is not in the usual standard form. A standard form can, however, be achieved via the additional linear transformation

$$(I + B_1)^{1/2} u_i = w_i \qquad u_i = (I + B_1)^{-1/2} w_i \qquad (i=1,2,...,n) \tag{29}$$

We introduce Eqs. (29) into Eq. (26) and premultiply the result by $(I + B_1)^{-1/2}$ to obtain

$$\lambda_i w_i = D w_i \qquad (i=1,2,...,n) \tag{30}$$

where

$$D \equiv (I + B_1)^{-1/2}(A_0 + A_1)(I + B_1)^{-1/2} \tag{31}$$

Introducing Eqs. (29) and (31) into Eqs. (28), we obtain orthonormality relations corresponding to Eq. (30), namely,

$$w_j^T w_i = \delta_{ij} \qquad w_j^T D w_i = \lambda_i\delta_{ij} \qquad (i,j=1,2,...,n) \tag{32}$$

Thus, we have two statements of the eigenvalue problem, Eqs. (26) and (30), and two statements of the orthonormality relations, Eqs. (28) and (32). In the following, we shall present derivations based upon both statements.

The Perturbed Eigenvalue Problem

Let us seek solutions to the eigenvalue problem (26) in the form of the expansions

$$\lambda_i = \lambda_{0i} + \lambda_{1i} + \lambda_{2i} + \ldots \tag{33a}$$

$$u_i = u_{0i} + u_{1i} + u_{2i} + \ldots \qquad (i=1,2,\ldots,n) \tag{33b}$$

where the first subscript of a term indicates the order of magnitude of that term. For example, λ_{1i} is of order one, and thus is one order of magnitude smaller than λ_{0i} and is one order of magnitude larger than λ_{2i}. Similarly, u_{2i} is one order of magnitude smaller than u_{1i} and is two orders of magnitude smaller than u_{0i}. We note that the expansions indicated in Eqs. (33) are not finite. In fact, if all terms in these expansions are determined, the expansions must converge to the exact eigensolutions of the perturbed eigenvalue problem. Because we shall determine and use only a finite number of terms, the resulting truncated expansions will provide merely approximate eigensolutions, which must tend to the unperturbed eigensolutions as B_1 and A_1 tend to zero. Furthermore, as B_1 and A_1 tend to zero, the terms in either of Eqs. (33) must maintain the same order of magnitude in relation to each other. For example, we cannot have the situation where λ_{2i} tends to zero as A_1 while λ_{1i} tends to zero as A_1^2.

The question arises as to what equations one should use to determine the perturbations to the unperturbed eigensolutions. Lancaster,[3] Franklin,[4] and Wilkinson[5] present derivations based upon the statement of the eigenvalue problem itself. Franklin carries the derivation only through first-order, while Lancaster and Wilkinson pursue the manipulations through second order. Wilkinson points out that normalization is lost and that the perturbed eigenvectors must be renormalized. To avoid this difficulty, we shall use the statement of orthonormality to determine the perturbations to the unperturbed eigensolutions. Furthermore, since our primary objective is to uncouple the equations of motion, Eq. (1), it is much more expenditious to use the equations indicating this uncoupling, namely, the orthonormality relations, Eqs. (28) or (32), or equivalently, Eqs. (4) and (5). We note that Eqs. (28) or (32) imply not only normalization but also satisfaction of the eigenvalue problem, Eqs. (26) or (30).

Introducing Eqs. (33) into Eqs. (28), and separating according to order of magnitude, we obtain

$$\mathcal{O}(0): \quad \begin{aligned} & u_{0j}^T u_{0i} = \delta_{ij} \\ & u_{0j}^T A_0 u_{0i} = \lambda_{0i}\delta_{ij} \end{aligned} \qquad (i,j=1,2,\ldots,n) \tag{34a}$$

$$\mathcal{O}(1): \quad \begin{aligned} & u_{0j}^T u_{1i} + u_{0j}^T B_1 u_{0i} + u_{1j}^T u_{0i} = 0 \\ & u_{0j}^T A_0 u_{1i} + u_{0j}^T A_1 u_{0i} + u_{1j}^T A_0 u_{0i} = \lambda_{1i}\delta_{ij} \end{aligned} \qquad (i,j=1,2,\ldots,n) \tag{34b}$$

$$\mathcal{O}(2): \quad \begin{aligned} & u_{0j}^T u_{2i} + u_{0j}^T B_1 u_{1i} + u_{1j}^T u_{1i} + u_{1j}^T B_1 u_{0i} + u_{2j}^T u_{0i} = 0 \\ & u_{0j}^T A_0 u_{2i} + u_{0j}^T A_1 u_{1i} + u_{1j}^T A_0 u_{1i} + u_{1j}^T A_1 u_{0i} \\ & \quad + u_{2j}^T A_0 u_{0i} = \lambda_{2i}\delta_{ij} \end{aligned} \qquad (i,j=1,2,\ldots,n) \tag{34c}$$

As anticipated, Eqs. (34a) are identical to Eqs. (24). Turning our attention to Eqs. (34b), we now wish to determine λ_{1i} and u_{1i}. Because u_{1i} is an n-vector in the space L^n, and because the vectors u_{0i} $(i=1,2,\ldots,n)$ span this space, we can represent u_{1i} as a linear combination of the u_{0i}. Hence, we can write

$$u_{1i} = \sum_{k=1}^{n} \epsilon_{ik} u_{0k} \qquad (i=1,2,\ldots,n) \tag{35}$$

and note that the smallness of u_{1i} relative to u_{0i} requires that the coefficients ϵ_{ik} be of order one. Substituting Eq. (35) into Eqs. (34b), and using Eqs. (34a), we obtain

$$\epsilon_{ij} + \epsilon_{ji} = -u_{0j}^T B_1 u_{0i} \qquad (i,j=1,2,\ldots,n) \tag{36a}$$

$$\lambda_{0j}\epsilon_{ij} + \lambda_{0i}\epsilon_{ji} = -u_{0j}^T A_1 u_{0i} + \lambda_{1i}\delta_{ij} \qquad (i,j=1,2,\ldots,n) \tag{36b}$$

When $i \neq j$, $\delta_{ij} = 0$, so that

$$\epsilon_{ij} = \frac{u_{0j}^T(-\lambda_{0i}B_1 + A_1)u_{0i}}{\lambda_{0i} - \lambda_{0j}} \qquad (i \neq j = 1,2,\ldots,n) \tag{37}$$

Alternatively, when $i=j$, Eqs. (36) become

$$2\epsilon_{ii} = -u_{0i}^T B_1 u_{0i} \qquad (i=1,2,\ldots,n) \tag{38a}$$

$$2\lambda_{0i}\epsilon_{ii} = -u_{0i}^T A_1 u_{0i} + \lambda_{1i} \qquad (i=1,2,\ldots,n) \tag{38b}$$

Solution of Eqs. (38) yields

$$\epsilon_{ii} = -\tfrac{1}{2} u_{0i}^T B_1 u_{0i} \qquad (i=1,2,\ldots,n) \tag{39}$$

$$\lambda_{1i} = u_{0j}^T(-\lambda_i B_1 + A_1)u_{0i} \qquad (i=1,2,\ldots,) \tag{40}$$

Equations (37), (39), and (40) determine the first-order perturbations λ_{1i} and u_{1i} fully. From Eq. (37), the motivation for our assumption that the unperturbed eigenvalues be distinct should be evident. We also note that λ_{1i} and u_{1i} do indeed tend to zero as B_1 and A_1 tend to zero. Introducing Eqs. (21) and (27) into Eq. (40), we can write

$$\lambda_{1i} = q_{0i}^T(-\lambda_{0i}M_1 + K_1)q_{0i} \qquad (i=1,2,\ldots,n) \tag{41}$$

Let us suppose for the moment that M_1 is a positive (negative) definite matrix. Then, bearing in mind that λ_{0i} $(i=1,2,\ldots,n)$ is a positive quantity, we observe that the presence of M_1 produces a decrease (increase) in the system's natural frequencies. Similarly, if matrix K_1 is positive (negative) definite, the natural frequencies are increased (decreased). This result fits nicely with the fact that the eigenvalues of oscillatory systems vary as stiffness divided by mass.

Let us now consider Eqs. (34c) and determine the second-order perturbations. Proceeding as before, we let

$$u_{2i} = \sum_{k=1}^{n} \bar{\epsilon}_{ik} u_{0k} \qquad (i=1,2,\ldots,n) \tag{42}$$

where $\bar{\epsilon}_{ik}$ is a second-order quantity. Substituting Eqs. (42) into Eqs. (34c), and utilizing Eqs. (34a), we obtain

$$\bar{\epsilon}_{ij} + \sum_{k=1}^{n} \epsilon_{ik} u_{0j}^T B_1 u_{0k} + \sum_{k=1}^{n} \epsilon_{ik}\epsilon_{jk} + \sum_{k=1}^{n} \epsilon_{jk} u_{0k}^T B_1 u_{0i} + \bar{\epsilon}_{ji} = 0 \qquad (i,j=1,2,\ldots,n) \tag{43a}$$

$$\lambda_{0j}\bar{\epsilon}_{ij} + \sum_{k=1}^{n} \epsilon_{ik} u_{0j}^T A_1 u_{0k} + \sum_{k=1}^{n} \lambda_{0k}\epsilon_{ik}\epsilon_{jk} + \sum_{k=1}^{n} \epsilon_{jk} u_{0k}^T A_1 u_{0i} + \lambda_{0i}\bar{\epsilon}_{ji} = \lambda_{2i}\delta_{ij} \qquad (i,j=1,2,\ldots,n) \tag{43b}$$

Using Eqs. (36), Eqs. (43) can be reduced to

$$\bar{\epsilon}_{ij} + \bar{\epsilon}_{ji} = \sum_{k=1}^{n} (\epsilon_{ik}\epsilon_{kj} + \epsilon_{ik}\epsilon_{jk} + \epsilon_{ki}\epsilon_{jk}) \qquad (i,j=1,2,\ldots,n) \tag{44a}$$

$$\lambda_{0j}\tilde{\epsilon}_{ij}+\lambda_{0i}\tilde{\epsilon}_{ji}=-(\lambda_{1j}\epsilon_{ij}+\lambda_{1i}\epsilon_{ji})+\lambda_{2i}\delta_{ij}+\sum_{k=1}^{n}(\lambda_{0j}\epsilon_{ik}\epsilon_{kj}+\lambda_{0k}\epsilon_{ik}\epsilon_{jk}+\lambda_{0i}\epsilon_{ki}\epsilon_{jk})\qquad(i,j=1,2,...,n)\tag{44b}$$

When $i\neq j$, $\delta_{ij}=0$, so that Eqs. (44) yield

$$\tilde{\epsilon}_{ij}=\frac{1}{\lambda_{0i}-\lambda_{0j}}\Big\{(\lambda_{1j}\epsilon_{ij}+\lambda_{1i}\epsilon_{ji})+\sum_{k=1}^{n}[(\lambda_{0i}-\lambda_{0j})\epsilon_{ik}\epsilon_{kj}+(\lambda_{0i}-\lambda_{0k})\epsilon_{ik}\epsilon_{jk}]\Big\}\qquad(i\neq j=1,2,...,n)\tag{45}$$

When $i=j$, Eqs. (44) become

$$2\tilde{\epsilon}_{ii}=\sum_{k=1}^{n}(\epsilon_{ik}^2+2\epsilon_{ik}\epsilon_{ki})\qquad(i=1,2,...,n)\tag{46a}$$

$$2\lambda_{0i}\tilde{\epsilon}_{ii}=-2\lambda_{1i}\epsilon_{ii}+\lambda_{2i}+\sum_{k=1}^{n}(\lambda_{0k}\epsilon_{ik}^2+2\lambda_{0i}\epsilon_{ik}\epsilon_{ki})\qquad(i=1,2,...,n)\tag{46b}$$

Solution of Eqs. (46) yields

$$\tilde{\epsilon}_{ii}=\tfrac{1}{2}\sum_{k=1}^{n}(\epsilon_{ik}^2+2\epsilon_{ik}\epsilon_{ki})\qquad(i=1,2,...,n)\tag{47}$$

$$\lambda_{2i}=2\lambda_{1i}\epsilon_{ii}+\sum_{k=1}^{n}(\lambda_{0i}-\lambda_{0k})\epsilon_{ik}^2\qquad(i=1,2,...,n)\tag{48}$$

Equations (45), (47), and (48) determine the second-order perturbations fully. Recalling that λ_{1i} and ϵ_{ij} are proportional to A_1 and B_1, we note that λ_{2i} and u_{2i} are proportional to quadratic products involving A_1 and/or B_1, and hence are second-order quantities.

In summary, we have

$$u_i=u_{0i}+\sum_{k=1}^{n}\epsilon_{ik}u_{0k}+\sum_{k=1}^{n}\tilde{\epsilon}_{ik}u_{0k}+...\qquad(i=1,2,...,n)\tag{49}$$

where the coefficients ϵ_{ik} and $\tilde{\epsilon}_{ik}$ are given by Eqs. (37), (39), (45), and (47). From the second of Eqs. (25), we can retrieve the eigenvectors of Eq. (2) in the form

$$q_i=L_0^{-T}u_i=L_0^{-T}\Big(u_{0i}+\sum_{k=1}^{n}\epsilon_{ik}u_{0k}+\sum_{k=1}^{n}\tilde{\epsilon}_{ik}u_{0k}+...\Big)\qquad(i=1,2,...,n)\tag{50}$$

Substitution of Eq. (50) into Eq. (16) yields the dynamic response $q(t)$.

For the purpose of comparing with later results, it is worthwhile to write the modal matrix

$$U=U_0+U_1+U_2...\tag{51}$$

where

$$U_1=U_0E\qquad U_2=U_0\tilde{E}\tag{52}$$

and where the elements of matrices E and $\tilde{E}$ are given by

$$E_{ki}=\epsilon_{ik}\qquad \tilde{E}_{ki}=\tilde{\epsilon}_{ik}\qquad(i,k=1,2,...,n)\tag{53}$$

Let us return to the second statement of the eigenvalue problem and orthonormality relations, given by Eqs. (30) and (32). Because the elements of matrix B_1 are assumed to be one order of magnitude smaller than unity, we can write

$$(I+B_1)^{-1/2}=I-\tfrac{1}{2}B_1+\tfrac{3}{8}B_1^2...\tag{54}$$

Then, introducing Eq. (54) into Eq. (31), we obtain

$$D=D_0+D_1+D_2+...\tag{55}$$

where

$$D_0=A_0\tag{56a}$$

$$D_1=\tfrac{1}{2}(A_1-A_0B_1)+\tfrac{1}{2}(A_1-A_0B_1)^T\tag{56b}$$

$$D_2=\tfrac{1}{8}(-4A_1+3A_0B_1+B_1A_0)B_1+\tfrac{1}{8}B_1(-4A_1+3A_0B_1+B_1A_0)^T\tag{56c}$$

Bearing in mind that matrices B_1, A_0, and A_1 are symmetric, we conclude from Eqs. (56) that matrices D_0, D_1, and D_2 are symmetric.

Once again, let us assume the expansions

$$\lambda_i=\lambda_{0i}+\lambda_{1i}+\lambda_{2i}+...\qquad(i=1,2,...,n)\tag{57a}$$

$$w_i=w_{0i}+w_{1i}+w_{2i}+...\qquad(i=1,2,...,n)\tag{57b}$$

Before delving into the perturbation procedure, it is instructive to inspect Eqs. (29) more closely. Introducing Eqs. (33b), (54), and (57b) into either of Eqs. (29), and equating terms of like order, we find that at order zero

$$w_{0i}=u_{0i}\qquad(i=1,2,...,n)\tag{58}$$

Furthermore, since $D_0=A_0$, we find that the zero-order orthonormality relations corresponding to Eqs. (32) are identical to Eqs. (24).

Substituting Eqs. (56) and (57) into Eqs. (32), and separating according to order of magnitude, we obtain

$$\mathcal{O}(0):\quad u_{0j}^Tu_{0i}=\delta_{ij},\quad u_{0j}^TA_0u_{0i}=\lambda_{0i}\delta_{ij}\qquad(i,j=1,2,...,n)\tag{59a}$$

$$\mathcal{O}(1):\quad u_{0j}^Tw_{1i}+w_{1j}^Tu_{0i}=0,\quad u_{0j}^TA_0w_{1i}+u_{0j}^TD_1u_{0i}+w_{1j}^TA_0u_{0i}=\lambda_{1i}\delta_{ij}\qquad(i,j=1,2,...,n)\tag{59b}$$

$$\mathcal{O}(2):\quad u_{0j}^Tw_{2i}+w_{1j}^Tw_{1i}+w_{2j}^Tu_{0i}=0,\quad u_{0j}^TA_0w_{2i}+u_{0j}^TD_1w_{1i}+u_{0j}^TD_2u_{0i}+w_{1j}^TA_0w_{1i}+w_{1j}^TD_1w_{0i}+w_{2j}^TA_0u_{0i}=\lambda_{2i}\delta_{ij}\qquad(i,j=1,2,...,n)\tag{59c}$$

As previously, Eqs. (59a) are already satisfied. In Eqs. (59b), we substitute

$$w_{1i}=\sum_{k=1}^{n}\gamma_{ik}u_{0k}\qquad(i=1,2,...,n)\tag{60}$$

Then, using Eqs. (59a), Eqs. (59b) can be written as

$$\gamma_{ij}+\gamma_{ji}=0\qquad(i,j=1,2,...,n)\tag{61a}$$

$$\lambda_{0j}\gamma_{ij}+\lambda_{0i}\gamma_{ji}=-u_{0j}^TD_1u_{0i}+\lambda_{1i}\delta_{ij}\qquad(i,j=1,2,...,n)\tag{61b}$$

For $i \neq j$, we obtain

$$\gamma_{ij} = -\gamma_{ji} = u_{0j}^T D_1 u_{0i} / (\lambda_{0i} - \lambda_{0j}) \qquad (i \neq j = 1,2,...,n) \qquad (62a)$$

while for $i = j$, we obtain

$$\gamma_{ii} = 0 \qquad (i = 1,2,...,n) \qquad (62b)$$

$$\lambda_{1i} = u_{0i}^T D_1 u_{0i} \qquad (i = 1,2,...,n) \qquad (63)$$

Turning our attention to Eqs. (59c), we let

$$w_{2i} = \sum_{k=1}^{n} \tilde{\gamma}_{ik} u_{0k} \qquad (i = 1,2,...,n) \qquad (64)$$

Upon substituting Eq. (64) into Eqs. (59c), and simplifying that result through the use of Eqs. (61), we obtain

$$\tilde{\gamma}_{ij} + \tilde{\gamma}_{ji} = -\sum_{k} \gamma_{ik}\gamma_{jk} \qquad (i,j = 1,2,...,n) \qquad (65a)$$

$$\lambda_{0j}\tilde{\gamma}_{ij} + \lambda_{0i}\tilde{\gamma}_{ji} = -u_{0j}^T D_2 u_{0i} - (\lambda_{1j} - \lambda_{1i})\gamma_{ij} + \lambda_{2i}\delta_{ij}$$
$$+ \sum_{k=1}^{n} (-\lambda_{0j} + \lambda_{0k} - \lambda_{0i})\gamma_{ik}\gamma_{jk} \qquad (i,j = 1,2,...,n) \qquad (65b)$$

For $i \neq j$, we obtain

$$\tilde{\gamma}_{ij} = \frac{1}{\lambda_{0i} - \lambda_{0j}} \Big\{ u_{0j}^T D_2 u_{0i} + (\lambda_{1j} - \lambda_{1i})\gamma_{ij}$$
$$+ \sum_{k=1}^{n} (\lambda_{0j} - \lambda_{0k})\gamma_{ik}\gamma_{jk} \Big\} \qquad (i \neq j = 1,2,...,n) \qquad (66a)$$

while for $i = j$, we obtain

$$\tilde{\gamma}_{ii} = -\tfrac{1}{2} \sum_{k=1}^{n} \gamma_{ik}^2 \qquad (i = 1,2,...,n) \qquad (66b)$$

$$\lambda_{2i} = u_{0i}^T D_2 u_{0i} + \sum_{k=1}^{n} (\lambda_{0i} - \lambda_{0k})\gamma_{ik}^2 \qquad (i = 1,2,...,n) \qquad (67)$$

Once again, we can summarize as follows:

$$w_i = w_{0i} + w_{1i} + w_{2i} + \ldots \qquad (i = 1,2,...,n) \qquad (68)$$

so that, using Eqs. (58), (60), and (64), we obtain

$$w_i = u_{0i} + \sum_{k=1}^{n} \gamma_{ik} u_{0k} + \sum_{k=1}^{n} \tilde{\gamma}_{ik} u_{0k} + \ldots \qquad (i = 1,2,...,n) \qquad (69)$$

From Eqs. (25), (29), and (54), we can retrieve the eigenvectors of Eq. (2) in the form

$$q_i = L_0^{-T} u_{0i} + L_0^{-T}\Big(\sum_{k=1}^{n} \gamma_{ik} u_{0k} - \tfrac{1}{2} B_1 u_{0i} \Big)$$
$$+ L_0^{-T}\Big(\sum_{k=1}^{n} \gamma_{ik} u_{0k} - \tfrac{1}{2} \sum_{k=1}^{n} \gamma_{ik} B_1 u_{0k} + \tfrac{3}{8} B_1^2 u_{0i} \Big) + \ldots$$
$$(i = 1,2,...,n) \qquad (70)$$

Substitution of Eq. (70) into Eq. (16) yields the dynamic response $q(t)$.

Once again, let us form the modal matrix

$$W = U_0 + W_1 + W_2 + \ldots \qquad (71)$$

where

$$W_1 = U_0 G \qquad W_2 = U_0 \tilde{G} \qquad (72)$$

and where the elements of matrices G and $\tilde{G}$ are given by

$$G_{ki} = \gamma_{ik} \qquad \tilde{G}_{ki} = \tilde{\gamma}_{ik} \qquad (i,k = 1,2,...,n) \qquad (73)$$

Use of Eqs. (29), (54), and (72) allows us to rewrite Eq. (71) as

$$U = U_0 + U_1 + U_2 + \ldots \qquad (74)$$

where

$$U_1 = U_0 G - \tfrac{1}{2} B_1 U_0 \qquad U_2 = U_0 \tilde{G} - \tfrac{1}{2} B_1 U_0 G + \tfrac{3}{8} B_1^2 U_0 \qquad (75)$$

We note the correspondence between Eqs. (51), (52) and (74), (75).

A comparison of the two perturbation procedures is in order. We note that for the first method, the matrix multiplications indicated in Eqs. (56) and (75) are not necessary. On the other hand, we note that in the second method, the effects of B_1 are included with those of A_1 in D_1. Thus, at first order one can say that $\gamma_{ij} = -\gamma_{ji}$ for the second method, while one can *not* say that $\epsilon_{ij} = -\epsilon_{ji}$ for the first method. The price paid for this simplification is the presence of matrix D_2 in the second method.

Numerical Example

Consider the two-degree-of-freedom spring-mass system of Fig. 1. The equations of motion can be written in the matrix form

$$M\ddot{q}(t) + Kq(t) = Q(t) \qquad (76)$$

where

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \qquad K = \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix} \qquad (77)$$

$$q(t) = \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} \qquad Q(t) = \begin{Bmatrix} Q_1(t) \\ Q_2(t) \end{Bmatrix} \qquad (78)$$

For the unperturbed mass and stiffness coefficients, let us take $m_1 = 1$ kg, $m_2 = 2$ kg, $k_1 = 200$ N/m, and $k_2 = 100$ N/m. Thus,

$$M_0 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad K_0 = \begin{bmatrix} 300 & -100 \\ -100 & 100 \end{bmatrix} \qquad (79)$$

For perturbations to the M_0 and K_0 matrices, let us take

$$M_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \qquad K_1 = \begin{bmatrix} 20 & -10 \\ -10 & 10 \end{bmatrix} \qquad (80)$$

The Cholesky decomposition matrix L_0 and its inverse are

$$L_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{bmatrix} \qquad L_0^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix} \qquad (81)$$

Following Eqs. (23) and (27), we can now construct the matrices

$$B_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.05 \end{bmatrix} \qquad A_0 = \begin{bmatrix} 300 & -50\sqrt{2} \\ -50\sqrt{2} & 50 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 20 & -5\sqrt{2} \\ -5\sqrt{2} & 5 \end{bmatrix} \qquad (82)$$

Table 1 Summary of exact, unperturbed and perturbed eigensolutions

	Exact	$\mathcal{O}(0)$	$\mathcal{O}(0)+\mathcal{O}(1)$	$\mathcal{O}(0)+\mathcal{O}(1)+\mathcal{O}(2)$
λ_1	32.138701	31.385934	32.205472	32.133597
λ_2	311.151342	318.614066	310.294528	311.241403
u_1	$\begin{bmatrix} 0.256815 \\ 0.939833 \end{bmatrix}$	$\begin{bmatrix} 0.254570 \\ 0.967054 \end{bmatrix}$	$\begin{bmatrix} 0.257383 \\ 0.938787 \end{bmatrix}$	$\begin{bmatrix} 0.256766 \\ 0.939871 \end{bmatrix}$
u_2	$\begin{bmatrix} 0.918225 \\ -0.262859 \end{bmatrix}$	$\begin{bmatrix} 0.967054 \\ -0.254570 \end{bmatrix}$	$\begin{bmatrix} 0.914611 \\ -0.263747 \end{bmatrix}$	$\begin{bmatrix} 0.918517 \\ -0.262802 \end{bmatrix}$

Table 2 Orthogonality check of eigenvectors

$U_0^T(I+B_1)U_0=$	$\begin{bmatrix} 1.053240 & 0.012309 \\ 0.012309 & 1.096760 \end{bmatrix}$
$(U_0+U_1)^T(I+B_1)(U_0+U_1)=$	$\begin{bmatrix} 0.998257 & -0.001037 \\ -0.001037 & 0.993204 \end{bmatrix}$
$(U_0+U_1+U_2)^T \times (I+B_1)(U_0+U_1+U_2)=$	$\begin{bmatrix} 1.000048 & 0.000077 \\ 0.000077 & 1.000560 \end{bmatrix}$

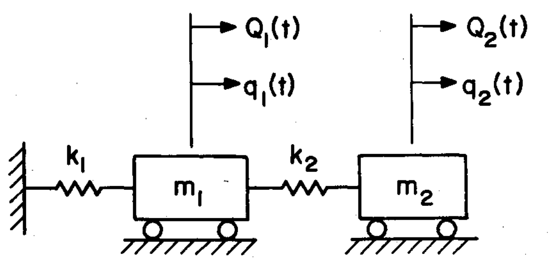

Fig. 1 The spring-mass system of the numerical example.

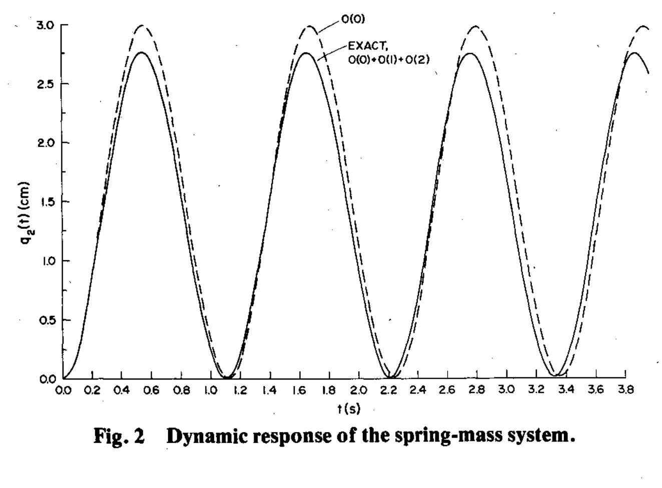

Fig. 2 Dynamic response of the spring-mass system.

Perturbations to the unperturbed eigensolutions were calculated as prescribed by the first method. The results are summarized in Table 1. Inspection of Table 1 reveals the quality of convergence. It is also instructive to inspect convergence via an orthonormality check of the perturbed eigenvectors. In Table 2, we present the matrix $U^T(I+B_1)U$ for $U=U_0$, $U=U_0+U_1$, and $U=U_0+U_1+U_2$.

The response to excitation in the form of a unit step function, applied to m_2, with amplitude 1 N has been computed. Figure 2 presents a plot of $q_2(t)$ vs t. Curves depicting the exact and $\mathcal{O}(0)$ responses are clearly distinct and are labeled as such. Within the accuracy of this plot, the $\mathcal{O}(0)+\mathcal{O}(1)$ and the $\mathcal{O}(0)+\mathcal{O}(1)+\mathcal{O}(2)$ responses are the same as the exact response.

The perturbations to the unperturbed eigensolutions were also computed via the second method. From Eqs. (56), we can construct

$$D_1=\begin{bmatrix} -10.0 & -1.767767 \\ -1.767767 & 2.5 \end{bmatrix} \quad D_2=\begin{bmatrix} 1.0 & 0.110485 \\ 0.110485 & -0.125 \end{bmatrix} \tag{83}$$

Results of these computations were identical to those using the first method. Thus, Tables 1 and 2 and Fig. 2 summarize the results for both methods.

Summary and Conclusions

A second-order perturbation analysis was developed for the algebraic eigenvalue problem

$$\lambda_i(M_0+M_1)q_i=(K_0+K_1)q_i \qquad (i=1,2,...,n)$$

where matrices M_0 and K_0 are real, symmetric, and positive definite, and where matrices M_1 and K_1 are real and symmetric. Based on the assumption that matrices M_1 and K_1 are of one order of magnitude smaller than M_0 and K_0, respectively, the analysis was developed in terms of the zero-order eigensolution, i.e., that associated with M_0 and K_0.

As an example, a two-degree-of-freedom spring-mass system was analyzed. Even for relatively large perturbations to the M_0 and K_0 matrices, results of the second-order perturbation analysis agree quite nicely with the exact solution, obtained by considering the perturbed system directly.

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